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Universality classes of the absorbing state transition in a system with interacting static and diffusive populations

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In this work, we study the critical behavior of a one-dimensional model that mimics the propagation of an epidemic process mediated by a density of diffusive individuals which can infect a static population upon contact. We simulate the above model on linear chains to determine the critical density of the diffusive population, above which the system achieves a statistically stationary active state, as a function of two relevant parameters related to the average lifetimes of the diffusive and nondiffusive populations. A finite-size scaling analysis is employed to determine the order parameter and correlation length critical exponents. For high-recovery rates, the critical exponents are compatible with the usual directed percolation universality class. However, in the opposite regime of low-recovery rates, the diffusion is a relevant mechanism responsible for the propagation of the disease and the absorbing state phase transition is governed by a distinct set of critical exponents.

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I. INTRODUCTION

Relevant features of several complex systems can be studied through the analysis of the critical behavior of nonequilibrium systems [1]. By complex system, we mean a classical stochastic many-particle system, defined by a given set of possible configurations, with transition rates governed by a master equation [2].The most interesting cases concern systems that are far away from thermal equilibrium. In equilibrium systems, the detailed balance condition is fulfilled, and the stationary probability distribution function is a uniform distribution (for isolated systems) or the Boltzmann-Gibbs distribution (for systems in contact with a thermal bath).

In this work, we are interested in second order phase transitions in nonequilibrium systems, mainly in systems possessing absorbing states, which are configurations that can be reached by the system, but it cannot be left. The simplest example of a model presenting a dynamic transition into absorbing states is the contact process (CP). It models the competition between two elementary processes in spatially distributed population. Self-replication is allowed as well as spontaneous annihilation of the entities. An example is a Markov process in which each site of a lattice presents two possible states, named *active* and *inactive*. Given a site i of the lattice, its state is said inactive if $S_i(t)=0$. If $S_i(t)=1$, the state is active. Depending on $S_i(t)$ and the number of active neighbors, a new value $S_i(t+dt)$ is assigned according certain rates. Also, the active state is not permanent, having a finite lifetime. Since only active sites at a given time t can activate sites at $t+dt$, the state with only inactive sites is an absorbing state. A short lifetime of active sites results in the whole system being driven to the absorbing state with only inactive sites. Above a critical lifetime, the system reaches a stationary active state with a fluctuating finite fraction of active sites [2–4].

The universality class of the CP is the Directed Percolation (DP) class. The DP universality class has been observed to describe a large class of models presenting a dynamic transition into a single absorbing state [5,6]. Even if the absorbing state presents more than one configuration, as in the pair contact process [7], the dynamic transition still belongs to the DP universality class, provided these configurations are not related to extra symmetries.

Of great interest in our work are the diffusive epidemic processes (DEP), which are reaction-diffusion-decay processes where two kinds of particles, say A and B, undergo the following reactions:

$$A+B \rightarrow 2B,$$

$$B \rightarrow A. \tag{1}$$

The presence of particle diffusion is an important mechanism that can influence the critical behavior of systems with absorbing states. This fact makes DEP an interesting model to study [8], as deviations from the DP class can be observed. Usually, A and B particles mimic healthy and sick individuals diffusing independently with diffusion constants D_A and D_B. Upon contact, sick individuals may infect healthy ones at a rate k_1. They also recover spontaneously at a rate k_2. Therefore, a competition between the contamination process (creation of B particles) and the recovery process (annihilation of B particles) takes place. For low concentrations of the average total population density ρ, the stationary state is characterized by a global extinction of the epidemics. Above a critical density ρ_c, there is a stable steady-state regime with a fluctuating finite density of sick individuals. Near ρ_c, the system exhibits a phase transition with the average density of

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sick individuals acting as the order parameter. The reaction-diffusion-decay process of Eq. (1) captures the essence of this model.

Above the critical dimension $d_c = 4$, fluctuations in the particle densities are irrelevant and the system is well-described by a mean-field approach. For lower dimensions corrections to the mean-field picture must be considered [2]. The critical properties of the stationary state for the special case of $D_A = D_B$ falls in the same universality class studied by Kree *et al.* in the context of a population density in a polluted environment [9]. Kree *et al.* computed the critical exponents to be $\eta = -\epsilon/8$ in first order of $\epsilon = 4 - d$, and $\nu = 2/d$ and $z = 2$ for all orders in ϵ. This process can also be characterized by the coupling between the fluctuations in the total density and the density of the species that is trying to survive. If fluctuations in the total density are suppressed, the transition falls in the universality class of DP. In a later work, van Wijland *et al.* [10] have shown that for $D_A < D_B$, the critical behavior is governed by a new fixed point. Within a Wilson renormalization group approach, they found that the transition falls in a new universality class with exponents given by $\nu = 2/d$, $\eta = 0$, $\beta = \nu(d + \eta)/2 = 1$, and $z = 2$ (in all orders in ϵ). In the opposite case of $D_A > D_B$, the renormalization group equations do not have a fixed point and they have conjectured the possible existence of a fluctuation-induced first-order transition in this regime. Numerical simulations in one-dimension, however, showed a continuous phase transition [11].

A generalized version of the pair contact process (PCP) with diffusion was investigated [12]. The authors presented a parameter related to the effect of memory, without which the model falls into the DP class. Long-term memory effects produce a distinct critical behavior that do not belong to the DP class. The critical exponents vary continuously with the parameter controlling the memory strength. In [13], a generalization of the bond directed percolation was studied, the infection rates decaying with $1/r^{d+\sigma}$, where d is the spatial dimension. It was observed that the critical exponents vary continuously with σ. The authors obtained results in good agreement with field theory, although the predicted transition to the DP class (when $\sigma = 2$) was not confirmed from the numerical simulations. A crossover from the directed percolation to the mean-field behavior has also been reported to occur in the diffusive contact process [14,15].

In previous works [16,17], some of the authors analyzed the critical behavior of one-dimensional DEP models with coupled static and diffusive populations. In [16], it was found that the dynamic transition of the considered model does not belong to the usual DP class. The results obtained are in agreement with previous field theoretical calculations showing that diffusion is an important mechanism that can influence the critical behavior of absorbing states phase transitions [9,10]. In [17], the role played by the static and diffusive populations was interchanged. The results obtained also strongly deviate from the DP class. The presence of coupled diffusive and nondiffusive populations thus offers a rich scenario that we explore in this work.

Within the above scenario, the main motivation of the present work is to analyze the role of diffusion in stochastic models with coupled diffusive and static populations. We study the critical behavior of a model that simulates the propagation of an epidemic process over a population mediated by a density of diffusive individuals which can infect a static population upon contact. We will employ a finite-size scaling scheme to locate the critical density and the relevant critical exponents for distinct regimes of the reaction rates. In particular, we will investigate the possible influence of the reaction rates on the critical behavior.

The paper is organized as follows: Sec. II presents our model and brings the mean-field result for the critical point. In Sec. III, we present our results for the critical behavior based on the finite-size scaling of data obtained from simulations in linear chains. Finally, we summarize and discuss our main results in Sec. IV.

II. MODEL

In what follows, we consider a model with two interacting species. One of the species corresponds to the individuals of a population that can be either in an inactive (noninfected) state or in an active (infected) state. These individuals occupy all sites of a one-dimensional lattice and are not allowed to diffuse. Sites in the active state have a finite lifetime, becoming inactive at a rate λ_p.

There is no direct contamination by contact between individuals. Therefore, without any mechanism to spread the infection, the population naturally evolves toward the vacuum state with no active sites. The spread of the disease will be considered to be mediated by a population of vectors which can be either in an active (infective) state or in an inactive (noninfective) state. The individuals of the vector population describe a random walk on the lattice. Vectors become noninfective at a rate λ_v. Active states are transmitted only between species whenever an inactive individual of one population occupies the same site of an active individual of the other population. The dynamics of the above described model can be represented by reaction-rate equation as

$$\begin{aligned} P_a + V_i &\xrightarrow{k_1} P_a + V_a, \\ P_i + V_a &\xrightarrow{k_2} P_a + V_a, \\ P_a &\xrightarrow{\lambda_p} P_i, \\ V_a &\xrightarrow{\lambda_v} V_i, \end{aligned} \tag{2}$$

where P_i, P_a, V_i, and V_a are the population densities of individuals and vectors in the inactive and active states. k_1 gives the infection rate of V_i upon contact with the active individuals P_a and k_2 stands for the rate at which the P_i population becomes infected by the vectors V_a. In the mean-field approach with uniformly distributed populations, the dynamics of the model is captured by the following set of ordinary differential equations:

$$\frac{\partial P_a}{\partial t} = k_2 P_i V_a - \lambda_p P_a,$$

$$\frac{\partial V_a}{\partial t} = k_1 P_a V_i - \lambda_v V_a,$$

$$P_a + P_i = 1,$$

$$V_a + V_i = \rho, \tag{3}$$

where ρ is the total density of the diffusive V population (V and P denote densities). In the above equations, the population of vector individuals are considered to be uniformly distributed over the lattice. Under this condition, the terms corresponding to the diffusion of the vector individuals do not influence the time evolution of the average population densities ($D\nabla^2 V_a = D\nabla^2 V_i = 0$). Both P_a and V_a recover spontaneously with rates λ_p and λ_v, respectively. For the stationary solution of Eq. (3), we obtain

$$P_a = \frac{k_1 k_2 \rho - \lambda_p \lambda_v}{\lambda_p k_1 + k_1 k_2 \rho}, \tag{4}$$

from which the critical density of the vector population is given by

$$\rho_c = \frac{\lambda_p \lambda_v}{k_1 k_2}, \tag{5}$$

which, for $\lambda_p = \lambda_v \equiv \lambda_{MF}$ and $k_1 = k_2 = k$, results in

$$\rho_c = \left(\frac{\lambda_{MF}}{k}\right)^2. \tag{6}$$

Defining λ as the probability per unit time that the active state P_a becomes inactive, one can relate λ with the mean-field reaction-rate λ_{MF}

$$P(a \to i) = \lambda \approx 1 - e^{-\lambda_{MF}} \tag{7}$$

or, equivalently,

$$\lambda_{MF} = \ln\left(\frac{1}{1-\lambda}\right), \tag{8}$$

which can be used to compare simulation results on a discrete time scale with the mean-field prediction. For small λ (low-recovery rates), the critical density for the spreading of the epidemic is low and one expect diffusion to be the main mechanism responsible for the propagation of the disease (since there will be few vector individuals on the lattice). Increasing λ will bring the system to a pure CP process, since in this case the critical vector density will be large and the disease will spread effectively by contact. In this sense, the transition may present features similar to the usual DP transition.

III. SIMULATIONS IN ONE DIMENSION

The above mean-field description, although giving a good qualitative description of the critical density dependence on the relevant reaction rates, is not able to capture the correct critical behavior of the absorbing state phase transition in low dimensions at which fluctuations in the population densities play a major role. In what follows, we are going to show results from simulations on finite linear chains of size L with periodic boundary conditions, thus discretizing both space and time. In each lattice sweep (considered as the time unit), each vector diffuses to one of its neighboring sites. No excluded volume interaction is considered. Therefore, a given site can support any number of vector individuals. Infective and noninfective vectors will be assumed to diffuse with the same diffusion rate. If a given vector individual is inactive, it becomes active whenever the site on which it is located is occupied by an individual in state P_a. On other hand, an active vector becomes inactive with probability ϕ. Within the same lattice sweep, each inactive individual becomes active if its site is occupied by at least one active vector. Active individuals become inactive with probability λ. Furthermore, since any finite system eventually becomes trapped in the vacuum state, we activate an individual chosen at random whenever the system becomes trapped. Note that λ and ϕ are related to λ_p and λ_v through Eqs. (7) and (8) and $k_1 = k_2 = 1$.

In the following, we will mainly report simulation data obtained from two sets of decay probabilities representing the regimes of low and high-recovery rates. In Fig. 1, we

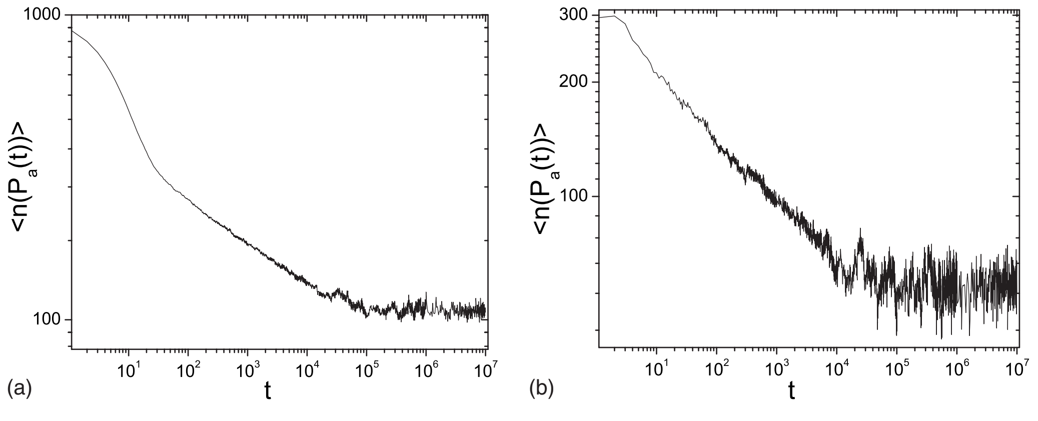

FIG. 1. Time evolution of the average number of infected individuals near criticality. Data are from a chain of 1000 sites. All individuals are considered to be initially in the active state. Time is measured in units of lattice sweeps. The relaxation time is estimated as the time needed for the system to reach the plateau region (typically 10^6 lattice sweeps for this system size). (a) Case $\phi = \lambda = 0.1$, $\rho_c = 0.116$, averaged over 100 independent runs. (b) Case $\phi = \lambda = 0.7$, $\rho_c = 25.89$, averaged over 20 runs.

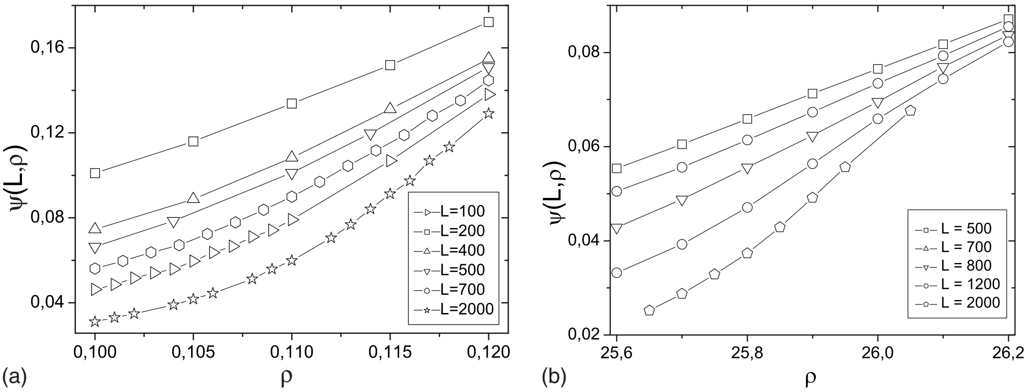

FIG. 2. Average stationary density of individuals in the active state versus the total density of vector particles ρ for distinct lattice sites. The stationary regime was considered to be achieved after $10L^2$ lattice sweeps. The average was performed considering 2×10^5 distinct configurations taken at each L lattice sweeps. Data were obtained from simulations with (a) $\phi=\lambda=0.1$ and (b) $\phi=\lambda=0.7$

show the temporal evolution of the average number of active individuals for (a) $\phi=\lambda=0.1$ (low-recovery rates) and (b) $\phi=\lambda=0.7$ (high-recovery rates). Data are averages over 100 distinct runs. These data were obtained using chains with 1000 sites and for vector densities near the absorbing state transitions, starting from a configuration with all individuals and vectors in the active state. This is the case on which the convergence to the stationary state becomes slower. In both cases, the stationary state is reached roughly after L^2 lattice sweeps. In the following, we disregarded the first $10L^2$ lattice sweeps for all chain sizes and vector's density to evaluate the average value of the order parameter density which is the average density of active individuals $\psi=\langle n(P_a)\rangle/L$ in the stationary regime as a function of the total density of vectors ρ. In order to start even closer to the stationary state, active and inactive individuals were initially distributed at random on the lattice sites.

In Fig. 2(a) ($\phi=\lambda=0.1$) and Fig. 2(b) ($\phi=\lambda=0.7$), we show the average stationary density of active individuals as a function of the total density ρ of vectors (average number of vector individuals per site), as obtained from simulations on lattices of distinct sizes. As $L\rightarrow\infty$, a transition from the vacuum to the active state takes place by increasing the vectors concentration. Notice that the critical vector density grows substantially with the recovery rate, in agreement with mean-field result.

In order to precisely locate the critical vector concentration, we measured the ratio between two moments of the density of infected individuals, defined as

$$m_L(\rho)=\frac{\langle[n(P_a)]^2\rangle}{[\langle n(P_a)\rangle]^2}. \tag{9}$$

The moment ratio $m_L(\rho)$ of large systems is known to be independent of the system size at the critical point [18,19]. In Fig. 3, we plot $m_L(\rho)$ as obtained from distinct lattice sizes, which allowed us to estimate the critical concentration $\rho_c=0.116(1)$ for $\phi=\lambda=0.1$ and $\rho_c=25.89(2)$ for $\phi=\lambda=0.7$. The critical moment ratio in the regime of high-recovery rates $[m^*=1.105(15)]$ is close to best estimated value found for the usual contact process transition [18]. The critical moment ratio for low-recovery rates is somewhat larger $[m^*=1.182(8)]$. It is interesting to stress that this value is similar to the one reported to hold for the case of the contact process transition with equally diffusing active and inactive particles [8].

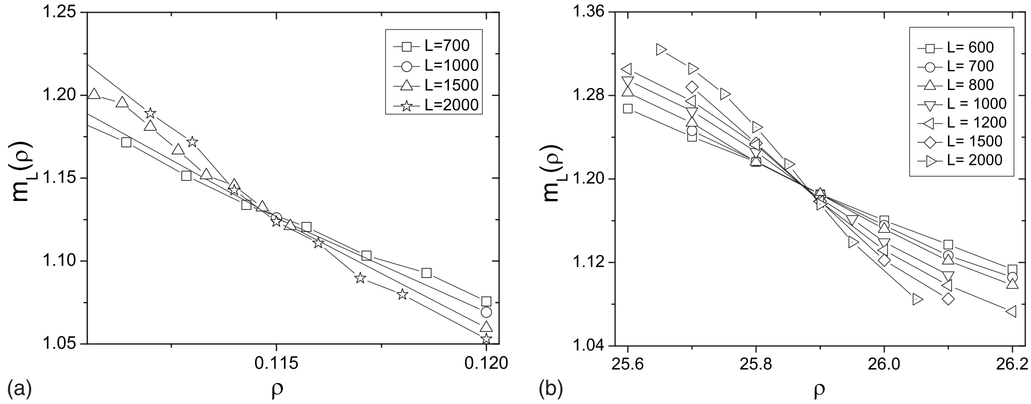

FIG. 3. (a) The moment ratio $m_L(\rho)=\langle[n(P_a)]^2\rangle/[\langle n(P_a)\rangle]^2$ as a function of the total density of vectors ρ for distinct lattice sizes. The scale invariance at the critical point allowed us to precisely estimate the critical vector density. (a) Case $\phi=\lambda=0.1$ for which our best estimate gives $\rho_c=0.116(1)$, $m_L(\rho_c=0.116)=1.105(15)$. (b) Case $\phi=\lambda=0.7$ for which we obtained $\rho_c=25.89(2)$, $m_L(\rho_c=25.89)=1.182(8)$.

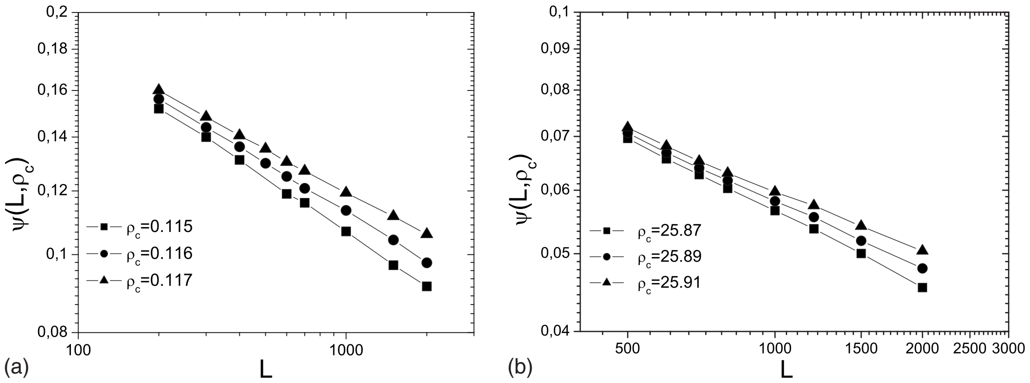

FIG. 4. (a) Finite-size scaling of the order parameter near the critical point. From the best fit to power laws, we estimate the critical exponents ratio β/ν. (a) Case $\phi=\lambda=0.1$ for which our estimate provided $\beta/\nu=0.20(2)$. (b) Case $\phi=\lambda=0.7$ for which $\beta/\nu=0.27(2)$. The error bar includes the error in the estimate of the critical density.

Once having located the critical concentration, finite-size scaling relations were used to compute the critical exponents characterizing such nonequilibrium phase transition. In particular, the critical order parameter density shall scale as $\psi(L,\rho_c)\propto L^{-\beta/\nu}$, while its logarithmic derivative scales as $\partial \ln \psi(L,\rho)/\partial\rho|_{\rho_c}\propto L^{1/\nu}$. These scaling laws are depicted in Figs. 4 and 5, respectively, from which we estimate $\beta/\nu=0.20(2)$, $\nu=1.99(11)$ for $\phi=\lambda=0.1$ and $\beta/\nu=0.27(2)$, $\nu=1.05(5)$ for case $\phi=\lambda=0.7$. We note that, although the exponent ratio β/ν is just weakly dependent of the recovery rate, the difference on ν is quite out of the error bar, indicating that the critical behaviors on these two regimes are distinct. In Fig. 6, we present data collapse of the order parameter density computed from different lattice sizes, using the estimated values of ρ_c, β/ν and ν. All data fall in a single curve in both cases of low and high-recovery rates, signaling the accuracy of the estimated critical parameters.

In Table I, we collect our estimates of ρ_c, β/ν, ν, and m^* for the two representative values of the recovery rate $\lambda=\phi$ considered. For high-recovery rates, our estimated exponents are similar to the ones exhibited by the usual contact process in one dimension. These correspond to the exponents of the directed percolation universality class in $(1+1)D$. In the limit of low-recovery rates, our results are consistent with the renormalization group prediction of $\nu=2/d=2$ for the diffusive epidemic process with equally diffusing active and inactive particles [10]. Although the renormalization group theory did not provide the exact value for β/ν, numerical simulations in one dimension have indeed estimated that this exponent ratio is smaller than the usual directed percolation value [8]. Furthermore, our estimate of the moment ratio in the case of low-recovery rates is consistent with the recently reported one for the diffusive epidemic process [8].

In order to understand the emergence of the diffusive epidemic process universality class, one shall notice that in the regime of low-recovery rates, individuals and vectors stay in the active state for a long time. In this case, the critical density necessary to sustain the epidemic is low and diffusion is, indeed, the relevant mechanism for the spreading of the disease. The observed continuous phase transition is in agreement with the renormalization group analysis of Ref. [10]. In the opposite case of high-recovery rates, the exponents are those of the standard directed percolation. In this regime, the density of vectors at the critical point is quite large but they stay a very short time in the active state. Therefore, the active state of a given individual can only be transmitted to neighboring individuals as it is the case in the direct contact process.

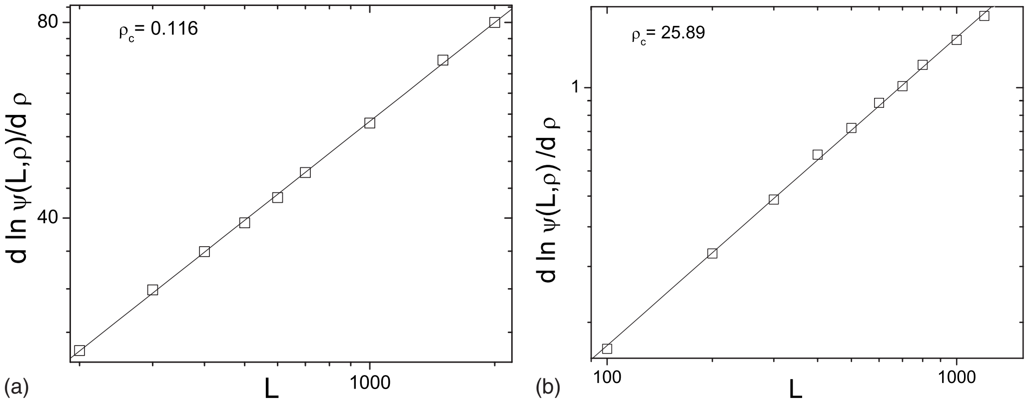

FIG. 5. Finite-size scaling of the logarithmic derivative of the order parameter at the critical point. From the best fit to power laws, we estimate the correlation length critical exponent. (a) $\phi=\lambda=0.1$ for which we obtained $\nu=1.99(11)$. (b) $\phi=\lambda=0.7$ for which $\nu=1.05(5)$. The error bars include the possible error in the estimates of the critical density.

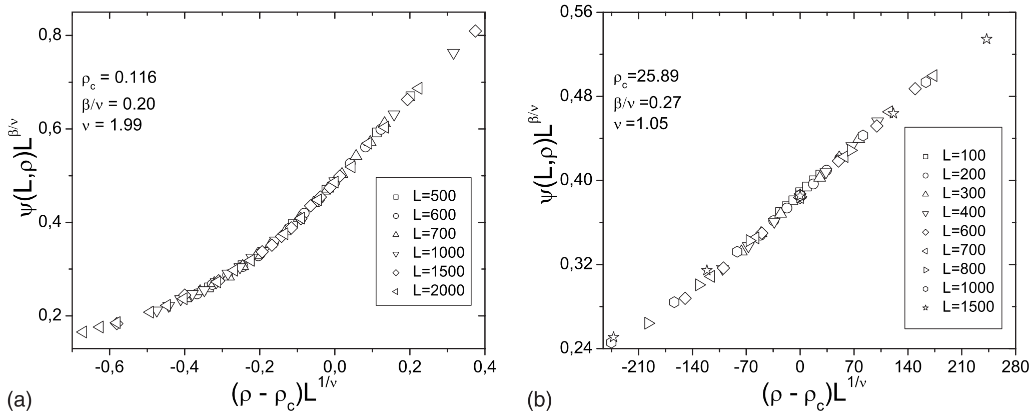

FIG. 6. Data collapse of the order parameter density computed from different linear lattice sizes L. (a) $\phi=\lambda=0.1$; (b) $\phi=\lambda=0.7$. Data from distinct chain sizes fall onto a single curve signaling the accuracy of the estimated critical quantities.

In Fig. 7, we show our results for the critical vector density for a series of additional intermediate values of the recovery rate $\lambda=\phi$, together with the mean-field prediction [Eq. (6) with $k=1$]. The estimated critical densities are substantially larger than the mean-field ones due to the relevant role played by order parameter fluctuations when the absorbing state transition takes place in low-dimensional systems. Concerning the critical behavior, our results indicate that, within the range of system sizes explored in our simulations, the exponents are compatible with the diffusive epidemic class universality class whenever the critical density is smaller or on the order of unity ($\phi=\lambda\lesssim 0.4$). On the other hand, the usual directed percolation universality class predominates for large critical densities ($\phi=\lambda\gtrsim 0.7$). In the regime of intermediate critical densities, the simulation data exhibit a crossover between these two universality classes, as illustrated in Fig. 8. However, simulations on much larger system sizes would be required in order to precisely locate the crossover point and to determine its properties. The case previously reported in Ref. [16] falls within this crossover regime. The lattice sizes considered in such preliminary work were much smaller than the presently reported ones, thus leading to less accurate estimates of the critical point and exponents. In particular, the order parameter exponent is known to be quite sensitive to the precise location of the critical point. This feature would explain the previously reported value of $\beta=0.34$ to be somewhat out of the interval here proposed. On the other hand, the correlation exponent ν is more robust and the reported value of $\nu=1.75$ seems to reflect the here reported crossover phenomenon.

TABLE I. Estimated critical density, critical exponents, and critical moment ratio for the cases of low- ($\lambda=\phi=0.1$) and high- ($\lambda=\phi=0.7$) recovery rates considered. The critical exponents and moment ratio of the usual one-dimensional contact process (CP-1D), as well as those for the diffusive epidemic process (DEP-1D), are also shown (Ref. [8,18]. These are compatible with our estimates for the high- and low-recovery rates regimes, respectively.

$\phi=\lambda$	ρ_c	β/ν	ν	m^*
0.1	0.116(1)	0.20(2)	1.99(11)	1.105(15)
0.7	25.89(2)	0.27(2)	1.05(5)	1.182(8)
CP-1D[a],[b]		0.252	1.09	1.1736(2)
DEP-1D[c]		0.192(4)	2.0(2)	1.093(10)

[a] Reference [20].
[b] Reference [18].
[c] Reference [8].

IV. CONCLUSIONS

We have investigated the critical behavior of an interacting two-species diffusion-limited reaction model which mimics the propagation of a disease in a static population mediated by a population of diffusive vectors. We showed that the proposed model presents a transition from an absorbing to an active state at a critical density of vectors. Using finite-size scaling, we computed some relevant critical exponents governing this nonequilibrium phase transition from simulations of this dynamical model in linear chains. We obtained that, for the parameters set employed, the scaling relations provided that the exponent ratio β/ν are just weakly dependent

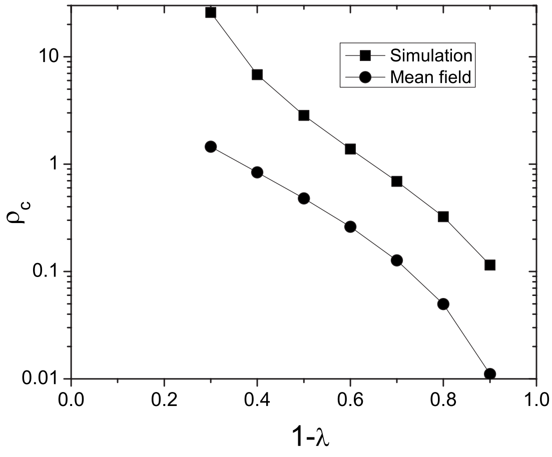

FIG. 7. Critical vector density versus $1-\lambda$. The corresponding mean-field values [Eq. (6) with $k=1$] are shown for comparison. The critical density obtained in the one-dimensional simulation exhibits a stronger divergence than the mean-field one as $\lambda\to 1$.

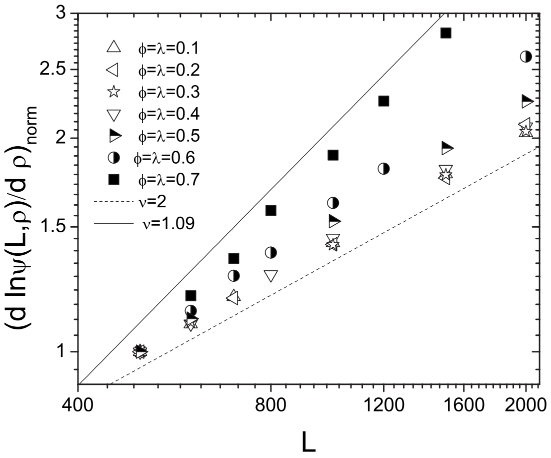

FIG. 8. Finite-size scaling of the logarithmic derivative of the order parameter at the critical point for several values of $\phi=\lambda$. The quantities are normalized in such a way that $\partial \ln \psi(L,\rho)/\partial\rho_{\mathrm{norm}}=1$ for $L=500$ in order to better visualize the dependence of the scaling behavior on the recovery rate. The curves corresponding to $\nu=2$ and $\nu=1.09$ are shown for comparison. The crossover from DEP to CP-1D can be clearly observed as $\phi=\lambda$ is increased.

of the characteristic transition rate from the active to the inactive state. On the other hand, the correlation length exponent ν was found to vary from the diffusive contact process universality class value for low-recovery rates to the direct percolation value for large recovery rates. The present results bring additional evidences that nonuniversal behavior can be a common feature of nonequilibrium phase transitions occurring in systems with diffusive populations. We hope the present work will stimulate larger scale simulations on models incorporating diffusive and nondiffusive populations that could probe the regime of intermediate reaction rates for which a crossover between the above universality classes shall take place.

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[1] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[2] H. Hinrichsen, Physica A **369**, 1 (2006).
[3] J. Marro and R. Dickman, *Nonequilibrium Phase Transictions in Lattice Models* (Cambridge University Press, Cambridge, England, 1999).
[4] R. Dickman, in *Nonequilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University, Cambridge, England, 1996).
[5] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[6] T. E. Harris, Ann. Probab. **2**, 969 (1974).
[7] I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993).
[8] D. S. Maia and R. Dickman, J. Phys.: Condens. Matter **19**, 065143 (2007).
[9] R. Kree, B. Schaub, and B. Schmittmann, Phys. Rev. A **39**, 2214 (1989).
[10] F. van Wijland, K. Oerding, and H. J. Hilhorst, Physica A **251**, 179 (1998).
[11] U. L. Fulco, D. N. Messias, and M. L. Lyra, Phys. Rev. E **63**, 066118 (2001).
[12] J. D. Noh and H. Park, Phys. Rev. E **69**, 016122 (2004).
[13] H. Hinrichsen and M. Howard, Eur. Phys. J. B **7**, 635 (1999).
[14] W. G. Dantas, M. J. de Oliveira, and J. F. Stilck, J. Stat. Mech.: Theory Exp. **2007**, P08009 (2007).
[15] A. Messer and H. Hinrichsen, J. Stat. Mech.: Theory Exp. **2008**, P04024 (2008).
[16] E. Macnadbay, R. Bezerra, U. L. Fulco, M. L. Lyra, and C. Argolo, Physica A **342**, 249 (2004).
[17] N. V. da Costa, U. L. Fulco, M. L. Lyra, and I. M. Gleria, Phys. Rev. E **75**, 031112 (2007).
[18] R. Dickman and J. K. L. da Silva, Phys. Rev. E **58**, 4266 (1998).
[19] S. Lübeck and R. D. Willmann, J. Stat. Phys. **115**, 1231 (2004).
[20] M. A. Muñoz, R. Dickman, A. Vespignani and S. Zapperi, Phys. Rev. E **59**, 6175 (1999).